Acta Sci. Math., 42 (1980), 205—210

On contractions of class $C_{1\cdot}$

PEI YUAN WU

It has been known that C_{11} contractions are quasi-similar to unitary operators. One may come to wonder what the corresponding result for the larger class of $C_{1\cdot}$ contractions is. Along this line SZ.-NAGY and FOIAŞ ([3], pp. 71—72) showed that an arbitrary $C_{1\cdot}$ contraction is a quasi-affine transform of an isometry. This result was also proved by DOUGLAS ([2], Lemma 4.5) using a different method. In the present paper we will refine the Sz.-Nagy and Foiaş technique more deeply to derive a "canonical" isometry for a completely non-unitary (c.n.u.) $C_{1\cdot}$ contraction whose defect indices are finite.

After we fix the notation and terminology in Section 1, we prove our main result in Section 2 in a series of lemmas. The notion of "multiplicity-free" $C_{1\cdot}$ contractions will be taken up in Section 3. We show that a c.n.u. multiplicity-free $C_{1\cdot}$ contraction with finite defect indices must be either of class C_{11} or of class C_{10}.

The author wishes to express his gratitude to Dr. L. Kérchy for pointing out some gap in, and simplifying the proof of, the main result in the preliminary version of this paper.

1. Preliminaries. A contraction T ($\|T\|\leqq 1$) is *completely non-unitary (c.n.u.)* if there exists no reducing subspace on which T is unitary. The *defect indices* of T are, by definition, $d_T=\operatorname{rank}(I-T^*T)^{1/2}$ and $d_{T*}=\operatorname{rank}(I-TT^*)^{1/2}$. $T\in C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n x\not\to 0$ (resp. $T^{*^n}x\not\to 0$) for all $x\neq 0$; $C_{11}=C_{1\cdot}\cap C_{\cdot 1}$. For every $C_{1\cdot}$ contraction T we have $d_T\leqq d_{T*}$. $T\in C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n x\to 0$ (resp. $T^{*^n}x\to 0$) for all x; $C_{10}=C_{1\cdot}\cap C_{\cdot 0}$.

Let $\mathbf{C}$ be the complex plane. For a positive integer n, let L_n^2 and H_n^2 denote the standard Lebesgue and Hardy spaces of $\mathbf{C}^n$-valued functions defined on the unit circle C. We will use "t" to denote the argument of a function defined on C and for

Received July 20, 1978, and in revised form October 15, 1979.

This research was partially supported by National Science Council of Taiwan.

analytic functions, we will freely identify $h(t)$ on the circle with its extension to the open unit disk $h(\lambda)$. If T is a c.n.u. contraction with defect indices $d_T=m$ and $d_{T^*}=n$, in the discussion of the following we shall consider its *functional model*, that is, we consider T being defined on $\mathfrak{H}\equiv[H_n^2\oplus\overline{\Delta L_m^2}]\ominus\{\Theta_T w\oplus\Delta w\colon w\in H_m^2\}$ by $T(f\oplus g)=P(e^{it}f\oplus e^{it}g)$ for $f\oplus g\in\mathfrak{H}$, where Θ_T denotes the characteristic function of T, $\Delta=(I-\Theta_T^*\Theta_T)^{1/2}$ and P denotes the (orthogonal) projection onto $\mathfrak{H}$. If Θ_T is the characteristic function of T, then the characteristic function of T^* is $\Theta_T^{\sim}$, where $\Theta_T^{\sim}(\lambda)=\Theta_T(\bar\lambda)^*$. For the details, the readers are referred to [3].

For arbitrary operators T_1, T_2 on $\mathfrak{H}_1$, $\mathfrak{H}_2$, respectively, $T_1\overset{i}{\prec}T_2$ denotes that T_1 is *injected into* T_2, that is, there exists an injection $X\colon\mathfrak{H}_1\to\mathfrak{H}_2$ such that $T_2X=XT_1$. If X also has dense range, then we say that X is a *quasi-affinity* and T_1 is a *quasi-affine transform* of T_2 (denoted by $T_1\prec T_2$). T_1, T_2 are *quasi-similar* $(T_1\sim T_2)$ if $T_1\prec T_2$ and $T_2\prec T_1$. For an arbitrary operator T on $\mathfrak{H}$, let μ_T denote the *multiplicity* of T, that is, the least cardinal number of a subset $\mathfrak{K}$ of elements in $\mathfrak{H}$ for which $\mathfrak{H}=\bigvee_{n=0}^{\infty}T^n\mathfrak{K}$. Note that if $T_1\overset{\sim}{\prec}T_2$ then $\mu_{T_2}\leqq\mu_{T_1}$.

2. $C_{1\cdot}$ contractions in general. Our purpose in this section is to prove the following main result.

Theorem 2.1. *Let T be a completely non-unitary $C_{1\cdot}$ contraction with defect indices $d_T=n\leqq d_{T^*}=m<\infty$. Then $T\prec S_{m-n}\oplus U$, where S_{m-n} denotes the unilateral shift on H_{m-n}^2 and U denotes the operator of multiplication by e^{it} on $\overline{\Delta L_n^2}$.*

If T is a $C_{1\cdot}$ contraction as above, then T^* is of class $C_{\cdot 1}$ and we may consider T^* being defined on $\mathfrak{H}\equiv[H_n^2\oplus\overline{\Delta^{\sim}L_m^2}]\ominus\{\Theta_T^{\sim}w\oplus\Delta^{\sim}w\colon w\in H_m^2\}$ by $T^*(f\oplus g)=P^{\sim}(e^{it}f\oplus e^{it}g)$ for $f\oplus g\in\mathfrak{H}$, where $\Delta^{\sim}=(I-\Theta_T^{\sim *}\Theta_T^{\sim})^{1/2}$ and $P^{\sim}$ denotes the (orthogonal) projection onto $\mathfrak{H}$. Let $P_1\colon\mathfrak{H}\to\overline{\Delta^{\sim}L_m^2}$ be the operator $P_1(f\oplus g)=g$ and let V be the operator of multiplication by e^{it} on $\overline{\Delta^{\sim}L_m^2}$. Then it is easily seen that $V^*P_1=P_1T$ and P_1 is injective (cf. [3], pp. 71—72). Thus $T\prec V^*|\overline{P_1\mathfrak{H}}$. What Lemmas 2.2, 2.3 and 2.4 below show is that $V^*|\overline{P_1\mathfrak{H}}$ is unitarily equivalent to $S_{m-n}\oplus U$.

Lemma 2.2. $\overline{P_1\mathfrak{H}}=\overline{\Delta^{\sim}L_m^2}\ominus\overline{\Delta^{\sim}\mathfrak{L}}$, *where* $\mathfrak{L}=\{f\in H_m^2\colon\Theta_T^{\sim}f=0\}$.

Proof. Let k be an element of $L_n^2\oplus\overline{\Delta^{\sim}L_m^2}$. We first show that $k\in\overline{\Delta^{\sim}L_m^2}\ominus\overline{P_1\mathfrak{H}}$ if and only if $k\perp L_n^2$ and $k\perp\mathfrak{H}$. Indeed, any $h\in\mathfrak{H}$ can be written in the form $h=f+g$, where $f\perp\overline{\Delta^{\sim}L_m^2}$ and $g\in\overline{P_1\mathfrak{H}}$. If k is orthogonal to any two of the elements h,f and g, then it is also orthogonal to the third element. Our assertion follows immediately. Since $L_n^2\oplus\overline{\Delta^{\sim}L_m^2}=(L_n^2\ominus H_n^2)\oplus\mathfrak{H}\oplus\{\Theta_T^{\sim}w\oplus\Delta^{\sim}w\colon w\in H_m^2\}$, the following three conditions are equivalent:

$$k \in \overline{\Delta^{\sim} L_m^2} \ominus \overline{P_1 \mathfrak{H}},$$

$$k \in \overline{\Delta^{\sim} L_m^2} \quad \text{and} \quad 0 \oplus k = \Theta_T^{\sim} w \oplus \Delta^{\sim} w \quad \text{for some} \quad w \in H_m^2,$$

$$k \in \overline{\Delta^{\sim} L_m^2} \quad \text{and} \quad k = \Delta^{\sim} w \quad \text{for some} \quad w \in \mathfrak{L},$$

which shows that $\overline{P_1 \mathfrak{H}} = \overline{\Delta^{\sim} L_m^2} \ominus \overline{\Delta^{\sim} \mathfrak{L}}$.

Lemma 2.3. *Let S_m and S_{m-n} denote the unilateral shifts on H_m^2 and H_{m-n}^2, respectively, and let $\mathfrak{L} = \{f \in H_m^2 \colon \Theta_T^{\sim} f = 0\}$. Then $S_m | \mathfrak{L} \cong S_{m-n}$.*

Proof. Since $\mathfrak{L}$ is an invariant subspace for S_m, $\mathfrak{L} = \Phi H_q^2$ for some inner function $\{\mathbf{C}^q, \mathbf{C}^m, \Phi\}$ where $q \leqq m$. Θ_T is $*$-outer implies that $\Theta_T^{\sim}$ is outer. Hence $\ker \Theta_T^{\sim}(t)$ has dimension $m-n$ for almost all t (cf. [3], p. 191), and it follows that $q \leqq m-n$.

On the other hand, considering the quotient field derived from the algebra H^{∞}, we see that the equation $\Theta_T^{\sim} f = 0$ has $m-n$ linearly independent solutions: $\psi_1, \ldots, \psi_{m-n}$. That is, $\psi_1, \ldots, \psi_{m-n} \in \mathfrak{L}$ and $\psi_1(t), \ldots, \psi_{m-n}(t)$ is a linearly independent system for almost all t (cf. [4], the proof of Theorem 5). Therefore, $m-n \leqq q$. Thus $q = m-n$ and the assertion follows.

Lemma 2.4. *$V^* | \overline{P_1 \mathfrak{H}}$ is unitarily equivalent to $S_{m-n} \oplus U$.*

Proof. Let $\mathfrak{L} = \Phi H_{m-n}^2$ be as in Lemma 2.3 and let $\psi_j = \Phi \eta_j$ for $j = 1, \ldots, m-n$, where η_j denotes the column vector with $m-n$ components whose j-th component is 1 and other components are 0. It is easily seen that for almost all t, $\psi_1(t), \ldots, \psi_{m-n}(t)$ are orthonormal eigenvectors of $\Delta^{\sim}(t)$ whose corresponding eigenvalues $\delta_1(t), \ldots, \delta_{m-n}(t)$ all constantly equal to 1. Since for almost all t, $\Delta^{\sim}(t)$ is a self-adjoint operator on $\mathbf{C}^m$ bounded by 0 and 1, we can extend $\{\psi_j(t)\}_1^{m-n}$ to an orthonormal base $\{\psi_j(t)\}_1^m$ of $\mathbf{C}^m$ consisting of eigenvectors of $\Delta^{\sim}(t)$, that is, such that $\Delta^{\sim}(t) \psi_j(t) = \delta_j(t) \psi_j(t)$, $j = 1, \ldots, m$, where the eigenvalues $\delta_j(t)$ are arranged in decreasing order:

$$1 = \delta_1(t) = \ldots = \delta_{m-n}(t) \geqq \delta_{m-n+1}(t) \geqq \ldots \geqq \delta_m(t) \geqq 0 \quad \text{a.e.}$$

Let $E_j = \{t \colon \operatorname{rank} \Delta^{\sim}(t) \geqq j\}$, $j = 1, \ldots, m$. Define $X \colon \overline{\Delta^{\sim} L_m^2} \to L^2(E_1) \oplus \ldots \oplus L^2(E_m)$ by $X(\Delta^{\sim} v) = x_1 \delta_1 \oplus \ldots \oplus x_m \delta_m$, where for any $v \in L_m^2$, $x_j(t) = (v(t), \psi_j(t))_{\mathbf{C}^m}$, $j = 1, \ldots, m$, and $(\ ,\ )_{\mathbf{C}^m}$ denotes the Euclidean inner product in $\mathbf{C}^m$. It was shown on pp. 272—273 of [3] that X can be extended to a unitary transformation from $\overline{\Delta^{\sim} L_m^2}$ onto $L^2(E_1) \oplus \ldots \oplus L^2(E_m)$ such that $XV = V'X$, where V' is the operator of multiplication by e^{it} on $L^2(E_1) \oplus \ldots \oplus L^2(E_m)$.

We complete the proof of this lemma in several steps. In each step the first statement is proved.

(i) $X\overline{\Delta^\sim \mathfrak{L}}=H^2_{m-n}\oplus\underbrace{0\oplus\dots\oplus 0}_{n}$. Let S_{m-n} and S_m denote the unilateral shifts on H^2_{m-n} and H^2_m, respectively. We have $\Phi S_{m-n}=S_m\Phi$. So $\bigvee_{i=0}^{\infty}\{S_m^i\psi_j, j=1, \dots, m-n\}=\mathfrak{L}$ and $\overline{\Delta^\sim\mathfrak{L}}=\bigvee_{i=0}^{\infty}\{\Delta^\sim S_m^i\psi_j,\ j=1, \dots, m-n\}=\bigvee_{i=0}^{\infty}\{V^i\Delta^\sim\psi_j,\ j=1, \dots, m-n\}$. Since X is a unitary operator for which $XV=V'X$, $X\overline{\Delta^\sim\mathfrak{L}}=\bigvee_{i=0}^{\infty}\{XV^i\Delta^\sim\psi_j,\ j=1, \dots, m-n\}=$ $=\bigvee_{i=0}^{\infty}\{V'^iX\Delta^\sim\psi_j,\ j=1, \dots, m-n\}=H^2_{m-n}\oplus 0\oplus\dots\oplus 0$, where in the last equation we used the relation $X\Delta^\sim\psi_j=\eta_j$ for $j=1, \dots, m-n$.

(ii) $V'^*|\underbrace{0\oplus\dots\oplus 0}_{m-n}\oplus L^2(E_{m-n+1})\oplus\dots\oplus L^2(E_m)$ *is unitarily equivalent to* U *on* $\overline{\Delta L^2_n}$. Let U be unitarily equivalent to the operator U' of multiplication by e^{it} on $L^2(F_1)\oplus\dots\oplus L^2(F_n)$, where $F_j=\{t\colon \operatorname{rank}\Delta(t)\geqq j\}$, $j=1, \dots, n$, are Borel subsets of C satisfying $F_1\supseteqq F_2\supseteqq\dots\supseteqq F_n$ (cf. [3], pp. 272—273). An elementary argument shows that $m+\operatorname{rank}\Delta(t)=n+\operatorname{rank}\Delta_*(t)=n+\operatorname{rank}\Delta^\sim(-t)$ a.e., where $\Delta_*=$ $=(I-\Theta_T\Theta_T^*)^{1/2}$. Hence $\operatorname{rank}\Delta(t)\geqq j$ if and only if $\operatorname{rank}\Delta^\sim(-t)\geqq m-n+j$. It follows that $F_j=E^\sim_{m-n+j}\equiv\{t\in C\colon -t\in E_{m-n+j}\}$, for $j=1, \dots, n$. We infer that U', hence U, is unitarily equivalent to $V'^*|0\oplus\dots\oplus 0\oplus L^2(E_{m-n+1})\oplus\dots\oplus L^2(E_m)$.

(iii) $V^*|\overline{P_1\mathfrak{H}}$ *is unitarily equivalent to* $S_{m-n}\oplus U$. By (i) and Lemma 2.2 we have $X^*[(L^2_{m-n}\ominus H^2_{m-n})\oplus L^2(E_{m-n+1})\oplus\dots\oplus L^2(E_m)]=\overline{\Delta^\sim L^2_m}\ominus\overline{\Delta^\sim\mathfrak{L}}=\overline{P_1\mathfrak{H}}$. Hence $V^*|\overline{P_1\mathfrak{H}}$ is unitarily equivalent to $V'^*|(L^2_{m-n}\ominus H^2_{m-n})\oplus L^2(E_{m-n+1})\oplus\dots\oplus L^2(E_m)$, which is, in term, unitarily equivalent to $S_{m-n}\oplus U$ by (ii).

This completes the proof.

We remark that from the proof above we can easily deduce that if T is a c.n.u. $C_{1\cdot}$ contraction with defect indices $d_T=n\leqq d_{T^*}=m<\infty$, and U, V and W denote the operators of multiplication by e^{it} on $\overline{\Delta L^2_n}$, $\overline{\Delta_* L^2_m}$ and L^2_{m-n}, respectively, then $V\cong W\oplus U$.

Note that the isometry of which T is a quasi-affine transform is, in general, not unique as is evident from the following lemma.

Lemma 2.5. *Let* S *and* U *be the unilateral and bilateral shifts on* H^2 *and* L^2, *respectively. Then* $S\prec U$.

Proof. Let g be an essentially bounded function in L^2 which is cyclic for U, that is, $L^2=\bigvee_{n=0}^{\infty}U^n g$ (cf. [5], proof of Lemma 4). Define $X\colon H^2\to L^2$ by $Xf=gf$ for $f\in H^2$. It is easily verified that X is a quasi-affinity intertwining S and U.

Corollary 2.6. *Let T be a C_{10} contraction with defect indices $d_T = n \leqq d_{T*} = m < \infty$. Then $T \prec S_{m-n}$.*

Proof. For a C_{10} contraction T we have $\Delta = (I - \Theta_T^* \Theta_T)^{1/2} = 0$. The assertion follows immediately from Theorem 2.1.

Actually, in the preceding situation Sz.-Nagy and Foiaş showed that T is completely injection-similar to the uniquely determined S_{m-n} (cf. [4]).

3. Multiplicity-free $C_1.$ contractions. A $C_1.$ contraction T is said to be *multiplicity-free* if it admits a cyclic vector, that is, $\mu_T = 1$. The following theorem gives equivalent conditions for multiplicity-free C_{10} contractions, which generalizes Proposition 2 of [4].

Theorem 3.1. *Let T be a C_{10} contraction with defect indices $d_T = n \leqq d_{T*} = m < \infty$ and let S denote the simple unilateral shift. Then the following are equivalent:*

(1) *T is multiplicity-free;*

(2) *$S \prec T$;*

(3) *$S \sim T$;*

(4) *$m - n = 1$ and there exists an $m \times 1$ matrix Δ over H^∞ such that $[\Delta, \Theta_T]$ is outer;*

(5) *$m - n = 1$ and there exist elements $x_1, \ldots, x_m$ in H^∞ such that*

$$x_1 \theta_1 - x_2 \theta_2 + \ldots + (-1)^{m+1} x_m \theta_m$$

is outer, where θ_j denotes the determinant of the $n \times n$ matrix obtained by deleting the j-th row from the matrix of Θ_T, $j = 1, \ldots, m$.

The proof essentially follows the same line of arguments as given by Sz.-Nagy and Foiaş [4] for the case $m = 2$, $n = 1$. We leave the verification to the readers.

Theorem 3.2. *Let T be a c.n.u. $C_1.$ contraction with defect indices $d_T = n \leqq d_{T*} = m < \infty$. Then the following are equivalent:*

(1) *T is multiplicity-free;*

(2) *either T is of class C_{10} and $T \sim S$ or T is of class C_{11} and $T \sim M_E$, where S denotes the simple unilateral shift and M_E denotes the operator of multiplication by e^{it} on $L^2(E)$ for some Borel subset $E \subseteq C$.*

Proof. (2)$\Rightarrow$(1). This is trivial since $\mu_T = \mu_S = \mu_{M_E} = 1$.

(1)$\Rightarrow$(2). By Theorem 2.1, $T \prec J \equiv S_{m-n} \oplus U$, where S_{m-n} denotes the unilateral shift on H^2_{m-n} and U denotes the operator of multiplication by e^{it} on $\overline{\Delta L^2_n}$. Thus (1) implies that $\mu_J \leqq \mu_T = 1$. It is an easy matter to check that either $J = S$

and T is of class C_{10} or $J=M_E$ for some Borel subset $E\subseteqq C$ and T is of class C_{11}. In the former case, $T\sim S$ follows from Theorem 3.1; in the latter, $T\sim M_E$ follows from Lemma 4.1 of [1], since T is itself quasi-similar to a unitary operator. This completes the proof.

References

[1] R. G. Douglas, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.
[2] R. G. Douglas, Canonical models, *Topics in operator theory*, Math. Surveys, No. 13, Amer. Math. Soc. (Providence, R. I., 1974), 161—218.
[3] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland — Akadémiai Kiadó (Amsterdam—Budapest, 1970).
[4] B. Sz.-Nagy and C. Foiaş, Jordan model for contractions of class $C_{\cdot 0}$, *Acta Sci. Math.*, **36** (1974), 305—322.
[5] P. Y. Wu, Jordan model for weak contractions, *Acta Sci. Math.*, **40** (1978), 189—196.

DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA

Current address:
DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47401, U.S.A.